Section 5.5

The Fundamental Theorem of Calculus II

(1) The Cumulative Area Function

(2) The Fundamental Theorem of Calculus

The Cumulative Area Function

Let f be a function and let a be a number. The **cumulative area function**

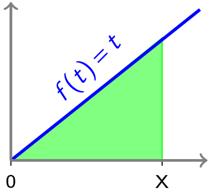
$$A_f(x) = \int_a^x f(t)\,dt$$

is the net area under the curve f on the interval $[a, x]$. (Note that this area depends on x.)

Example: Let $f(t) = t$ and $a = 0$. Then

$$A_f(x) = \int_0^x t\,dt = x^2/2$$

$=$ area of a triangle with base x and height x.



Remark on notation: $A_f(x)$ is a function of x, not of t. The letter t is just a "dummy variable" that has no meaning outside the integral.

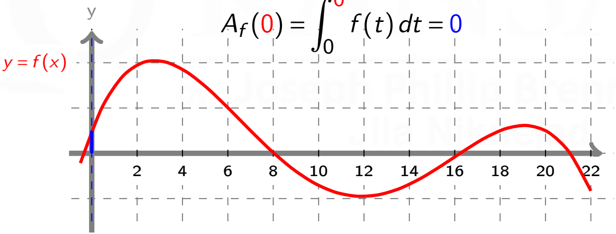
The Cumulative Area Function

Let f be a function and let a be a number. The **cumulative area function**

$$A_f(x) = \int_a^x f(t)\,dt$$

is the net area under the curve f on the interval $[a, x]$.

$$A_f(0) = \int_0^0 f(t)\,dt = 0$$



The Fundamental Theorem of Calculus, Part II

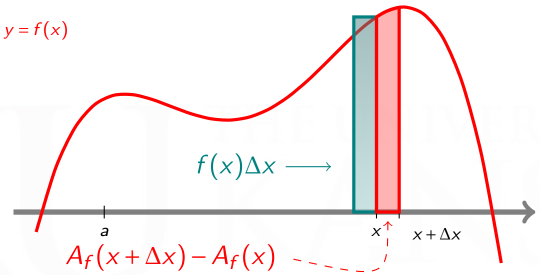
Graph of $f(t) \frac{d}{dx} A_f(x)$	**Area function** $A_f(x) = \int_a^x f(t)\,dt$
Above the x-axis Below the x-axis	Increasing Decreasing
Zero	Local extremum
Increasing Decreasing	Concave up Concave down

The Fundamental Theorem of Calculus II (FTC-2)

Suppose f is continuous on the interval $[a, b]$. Then, for all x in $[a, b]$:

$$\frac{d}{dx}(A_f(x)) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

The Idea Behind FTC-2



As Δx gets smaller and smaller, the average height of the red strip approaches $f(x)$. Therefore:

$$f(x) = \lim_{\Delta x \to 0} \frac{A_f(x+\Delta x) - A_f(x)}{\Delta x} = \frac{d}{dx} A_f(x) = \frac{d}{dx} \int_0^x f(t)\, dt$$

The Fundamental Theorem of Calculus II (FTC-2)

Suppose f is continuous on the interval $[a, b]$. Then, for all x in $[a, b]$:

$$\frac{d}{dx}\left(A_f(x)\right) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

Example 1a: $\dfrac{d}{dx}\left(\displaystyle\int_x^a f(t)\,dt\right) = \dfrac{d}{dx}\left(-\displaystyle\int_a^x f(t)\,dt\right) = -f(x).$

(Here the area measured by $\displaystyle\int_x^a f(t)\,dt$ gets <u>smaller</u> as x increases.)

Example 1b: $\dfrac{d}{dx}\displaystyle\int_a^{3x^2} f(t)\,dt = 6x\,f(3x^2)$ by the Chain Rule.

More generally, $\dfrac{d}{dx}\displaystyle\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x))\,g'(x) - f(h(x))\,h'(x).$

The Fundamental Theorem of Calculus

Let $f(x)$ be continuous on $[a,b]$ and let F be an antiderivative of f.

Let $A_f(x) = \int_a^x f(t)\,dt$. Then:

(FTC Part I) $\int_a^b f(x)\,dx = F(b) - F(a)$.

(FTC Part II) $\frac{d}{dx}(A_f(x)) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$.

The Fundamental Theorem of Calculus shows that integration and differentiation are **inverse operations**.

- If you start with a continuous function f and form the integral $\int_a^x f(t)\,dt$, then you get back the original function by differentiating:

$$f(x) \quad \rightarrowtail \text{Integrate} \rightarrowtail \quad \int_a^x f(t)\,dt$$

$$\rightarrowtail \text{Differentiate} \rightarrowtail \quad \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

- If you differentiate a function f and then integrate it, then you get back the original function, up to a constant:

$$f(x) \quad \rightarrowtail \text{Differentiate} \rightarrowtail \quad \frac{d}{dx}(f(x)) = f'(x)$$

$$\rightarrowtail \text{Integrate} \rightarrowtail \quad \int_a^x f'(t)\,dt = f(x) - f(a)$$